Bicomplexes, Integrable Models, and Noncommutative Geometry

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Abstract

We discuss a relation between bicomplexes and integrable models, and consider corresponding noncommutative (Moyal) deformations. As an example, a noncommutative version of a Toda field theory is presented.

1 Introduction

Soliton equations and integrable models are known to possess a vanishing curvature formulation depending on a parameter, say λ (cf [1], for example). This geometric formulation of integrable models is easily extended [2, 3] to generalized geometries, in particular in the sense of noncommutative geometry where, on a basic level, the algebra of differential forms (over the algebra of smooth functions) on a manifold is generalized to a differential calculus over an associative (and not necessarily commutative) algebra $\mathcal{A}$.

A bicomplex associated with an integrable model is a special case of a zero curvature formulation. More precisely, let $\mathcal{M} = \bigoplus_{r \geq 0} \mathcal{M}^r$ be an $\mathbb{N}_0$-graded linear space (over $\mathbb{R}$ or $\mathbb{C}$) and $\mathrm{d}, \delta : \mathcal{M}^r \to \mathcal{M}^{r+1}$ two linear maps satisfying[1]

$$\mathrm{d}^2 = 0 \,, \quad \delta^2 = 0 \,, \quad \mathrm{d}\,\delta + \delta\,\mathrm{d} = 0 \tag{1.1}$$

(typically as a consequence of certain field equations). Then $(\mathcal{M}, \mathrm{d}, \delta)$ is called a *bicomplex*. Special examples are bi-differential calculi [3]. However, we do not need d and δ to be graded derivations (into some bimodule), i.e., they do not have to satisfy the Leibniz rule.

Given a bicomplex, there is an iterative construction of "generalized conserved densities" in the sense of δ-closed elements of the bicomplex (see section 2). In some examples they reproduce directly the conserved quantities of an integrable model. In other examples, the

[1] In terms of $\mathrm{d}_\lambda = \delta - \lambda\,\mathrm{d}$ with a constant λ, the three bicomplex equations are combined into the single zero curvature condition $\mathrm{d}_\lambda^2 = 0$ (for all λ).

relation is less direct. Anyway, the existence of such a chain of δ-closed elements is clearly a distinguished feature of the model with which the bicomplex is associated.

Noncommutative examples are in particular obtained by starting with a classical integrable model, deforming an associated bicomplex by replacing the ordinary product of functions with the Moyal $*$-product and thus arriving at a noncommutative model. As an example, a noncommutative extension of a Toda field theory is considered in section 3. Field theory on noncommutative spaces has gained more and more interest during the last years. A major impulse came from the discovery that a noncommutative gauge field theory arises in a certain limit of string, D-brane and M theory (see [5] and the references cited there). We also refer to [6] for some work on Moyal deformations of integrable models.

2 The bicomplex linear equation

Let us assume that, for some $s \in \mathbb{N}$, there is a (nonvanishing) $\chi^{(0)} \in \mathcal{M}^{s-1}$ with $\mathrm{d}J^{(0)} = 0$ where $J^{(0)} = \delta\chi^{(0)}$. Defining $J^{(1)} = \mathrm{d}\chi^{(0)}$, we have $\delta J^{(1)} = -\mathrm{d}\delta\chi^{(0)} = 0$ using (1.1). If the δ-closed element $J^{(1)}$ is δ-exact, this implies $J^{(1)} = \delta\chi^{(1)}$ with some $\chi^{(1)} \in \mathcal{M}^{s-1}$. Next we define $J^{(2)} = \mathrm{d}\chi^{(1)}$. Then $\delta J^{(2)} = -\mathrm{d}\delta\chi^{(1)} = -\mathrm{d}J^{(1)} = -\mathrm{d}^2\chi^{(0)} = 0$. If the δ-closed element $J^{(2)}$ is δ-exact, then $J^{(2)} = \delta\chi^{(2)}$ with some $\chi^{(2)} \in \mathcal{M}^{s-1}$. This can be iterated further and leads to a possibly infinite chain (see the figure below) of elements $J^{(m)}$ of $\mathcal{M}^s$ and $\chi^{(m)} \in \mathcal{M}^{s-1}$ satisfying

$$J^{(m+1)} = \mathrm{d}\chi^{(m)} = \delta\chi^{(m+1)} \,. \tag{2.1}$$

More precisely, the above iteration continues from the mth to the $(m+1)$th level as long as $\delta J^{(m)} = 0$ implies $J^{(m)} = \delta\chi^{(m)}$ with an element $\chi^{(m)} \in \mathcal{M}^{s-1}$. Of course, there is no obstruction to the iteration if $H^s_\delta(\mathcal{M})$ is trivial, i.e., when all δ-closed elements of $\mathcal{M}^s$ are δ-exact. But in general the latter condition is too strong, though in several examples it can indeed be easily verified [3]. Introducing

$$\chi = \sum_{m \geq 0} \lambda^m \, \chi^{(m)} \tag{2.2}$$

with a parameter λ, the essential ingredients of the above iteration procedure are summarized in the *linear equation* associated with the bicomplex:

$$\delta(\chi - \chi^{(0)}) = \lambda \, \mathrm{d} \, \chi \,. \tag{2.3}$$

$$\begin{tikzcd}
& \chi^{(0)} \arrow[dl, "\delta"'] \arrow[dr, "d"] & & \chi^{(1)} \arrow[dl, "\delta"'] \arrow[dr, "d"] & & \chi^{(2)} \arrow[dl, "\delta"'] \arrow[dr, "d"] & & \\
J^{(0)} \arrow[dr, "d"'] & & J^{(1)} \arrow[dl, "\delta"] \arrow[dr, "d"'] & & J^{(2)} \arrow[dl, "\delta"] \arrow[dr, "d"'] & & J^{(3)} \arrow[dl, "\delta"] & \cdots \\
& 0 & & 0 & & 0 & &
\end{tikzcd}$$

Given a bicomplex, we may start with the linear equation (2.3). Let us assume that it admits a (non-trivial) solution χ as a (formal) power series in λ. The linear equation then leads to $\delta\chi^{(m)} = \mathrm{d}\chi^{(m-1)}$. As a consequence, the $J^{(m+1)} = \mathrm{d}\chi^{(m)}$ are δ-exact. Therefore, even if the cohomology $H^s_\delta(\mathcal{M})$ is *not* trivial, the solvability of the linear equation ensures that the δ-closed $J^{(m)}$ appearing in the iteration are δ-exact.

In all the examples which we presented in [3, 4], the bicomplex space can be chosen as $\mathcal{M} = \mathcal{M}^0 \otimes \Lambda$ where $\Lambda = \bigoplus_{r=0}^n \Lambda^r$ is the exterior algebra of an n-dimensional vector space with a basis ξ^r, $r = 1, \ldots, n$, of Λ^1. It is then sufficient to define the bicomplex maps d and δ on $\mathcal{M}^0$ since via

$$\mathrm{d}\Big(\sum_{i_1,\ldots,i_r=1}^{n} \phi_{i_1\ldots i_r}\, \xi^{i_1}\cdots\xi^{i_r}\Big) = \sum_{i_1,\ldots,i_r=1}^{n} (\mathrm{d}\phi_{i_1\ldots i_r})\, \xi^{i_1}\cdots\xi^{i_r} \tag{2.4}$$

(and correspondingly for δ) they extend as linear maps to the whole of $\mathcal{M}$.

3 Noncommutative deformation of a Toda model

The $*$-product on the space $\mathcal{F}$ of smooth functions of two coordinates x and t is given by

$$f * h = m \circ e^{\theta P/2}(f \otimes h) = fh + \frac{\theta}{2}\,\{f, h\} + \mathcal{O}(\theta^2) \tag{3.1}$$

where θ is a parameter, $m(f \otimes h) = fh$ and $P = \partial_t \otimes \partial_x - \partial_x \otimes \partial_t$. Furthermore, $\{\,,\,\}$ is the Poisson bracket, i.e., $\{f, h\} = (\partial_t f)\,\partial_x h - (\partial_x f)\,\partial_t h$. For the calculations below it is helpful to notice that partial derivatives are derivations of the algebra $(\mathcal{F}, *)$.

A bicomplex associated with an integrable model can be deformed by replacing the ordinary product of functions with the noncommutative $*$-product. This then induces a deformation of the integrable model with very special properties since the iterative construction of generalized conservation laws still works. As a specific example, we construct a noncommutative extension of the Toda field theory on an open finite one-dimensional lattice. Other examples can be obtained in the same way.

Let us start from the trivial bicomplex which is determined by

$$\delta\phi = (\partial_t - \partial_x)\phi\,\xi^1 + (S - I)\phi\,\xi^2\,, \quad \mathrm{d}\phi = -S^T\phi\,\xi^1 + (\partial_t + \partial_x)\phi\,\xi^2 \tag{3.2}$$

where ϕ is a vector with n components (which are functions) and S^T the transpose of

$$S = \sum_{i=1}^{n-1} E_{i,i+1}\,, \qquad (E_{i,j})^k{}_l = \delta^k_i\,\delta_{j,l}\,. \tag{3.3}$$

Let G be an $n \times n$ matrix of functions which is invertible in the sense $G^{-1} * G = I$ where I is the $n \times n$ unit matrix. Now we introduce a "dressing" for d:

$$\mathrm{D}\phi = G^{-1} * \mathrm{d}(G * \phi) = -(L * \phi)\,\xi^1 + (\partial_t + \partial_x + M*)\phi\,\xi^2 \tag{3.4}$$

where

$$L = G^{-1} * S^T * G\,, \quad M = G^{-1} * (G_t + G_x)\,. \tag{3.5}$$

Note that $\mathrm{D}^2\phi = G^{-1} * \mathrm{d}^2(G * \phi) = 0$. The only nontrivial bicomplex equation is $\delta \mathrm{D} + \mathrm{D}\delta = 0$ which reads

$$M_t - M_x = L * S - S * L \; . \tag{3.6}$$

Hence, if this equation holds, then $(\mathcal{F}^n \otimes \Lambda, \mathrm{D}, \delta)$ is a bicomplex. Let us now choose

$$G = \sum_{i=1}^{n} G_i \, E_{ii} \tag{3.7}$$

with functions G_i for which the invertibility assumption requires $G_i^{-1} * G_i = 1$. Then

$$L = \sum_{i=1}^{n-1} G_{i+1}^{-1} * G_i \, E_{i+1,i} \, , \quad M = \sum_{i=1}^{n} M_i \, E_{ii} \, , \quad M_i = G_i^{-1} * (\partial_t + \partial_x) G_i \; . \tag{3.8}$$

Writing

$$G_i = e^{q_i}(1 + \theta \, \tilde{q}_i) + \mathcal{O}(\theta^2) \tag{3.9}$$

we have $G_i^{-1} = e^{-q_i}(1 - \theta \, \tilde{q}_i) + \mathcal{O}(\theta^2)$ and it follows from (3.6) that the functions q_i have to solve the Toda field theory equations

$$\begin{aligned} (\partial_t^2 - \partial_x^2) q_i &= e^{q_{i-1} - q_i} - e^{q_i - q_{i+1}} \qquad i = 2, \ldots, n-1 \\ (\partial_t^2 - \partial_x^2) q_1 &= -e^{q_1 - q_2} \, , \quad (\partial_t^2 - \partial_x^2) q_n = e^{q_{n-1} - q_n} \; . \end{aligned} \tag{3.10}$$

Furthermore, the functions $\tilde{q}_i$ are subject to the following linear equations,

$$\begin{aligned} (\partial_t^2 - \partial_x^2) \tilde{q}_1 &= \{\partial_t q_1, \partial_x q_1\} - e^{q_1 - q_2}(\tilde{q}_1 - \tilde{q}_2) \\ (\partial_t^2 - \partial_x^2) \tilde{q}_i &= \{\partial_t q_i, \partial_x q_i\} + e^{q_{i-1} - q_i}(\tilde{q}_{i-1} - \tilde{q}_i) - e^{q_i - q_{i+1}}(\tilde{q}_i - \tilde{q}_{i+1}) \\ (\partial_t^2 - \partial_x^2) \tilde{q}_n &= \{\partial_t q_n, \partial_x q_n\} + e^{q_{n-1} - q_n}(\tilde{q}_{n-1} - \tilde{q}_n) \; . \end{aligned} \tag{3.11}$$

A 1-form $J = P \, \xi^1 + R \, \xi^2$ is δ-closed iff $(\partial_t - \partial_x) R = (S - I) \, P$. For $J = \lambda \, \mathrm{D}\chi$ (cf (2.1)) we have $P = -\lambda \, L * \chi$ and $R = \lambda \, (\partial_t + \partial_x + M*) \chi$ and thus

$$\partial_t [\lambda \, (\partial_t + M_i *) \chi_i] = \partial_x [\lambda \, (\partial_x + M_i *) \chi_i] + P_{i+1} - P_i \tag{3.12}$$

$(i = 1, \ldots, n)$ where we have to set $P_{n+1} = 0$. Using $P_1 = 0$, we find that

$$Q = \lambda \int dx \, \sum_{i=1}^{n} \left(\partial_t \chi_i + G_i^{-1} * [(\partial_t + \partial_x) G_i] * \chi_i \right) \tag{3.13}$$

is conserved, i.e., $dQ/dt = 0$, provided that the expressions $\partial_x \chi_i + M_i * \chi_i$ vanish at $x = \pm\infty$. In order to further evaluate this expression, we have to explore the linear system associated with the bicomplex. Choosing $\chi^{(0)} = \sum_{i=1}^{n} e_i$, where e_i is the vector with components $(e_i)_j = \delta_{ij}$, we have $\delta \chi^{(0)} = -e_n \, \xi^2$ and $\mathrm{D}\delta \chi^{(0)} = 0$. Now we find

$$\begin{aligned} Q^{(1)} &= \int dx \, \sum_{i=1}^{n} M_i = \int dx \, \sum_{i=1}^{n} G_i^{-1} * (\partial_t + \partial_x) G_i \\ &= \int dx \, \sum_{i=1}^{n} \partial_t q_i + \theta \int dx \, \sum_{i=1}^{n} \left((\partial_t + \partial_x) \tilde{q}_i + \frac{1}{2} \{(\partial_t + \partial_x) q_i, q_i\} \right) + \mathcal{O}(\theta^2) \end{aligned} \tag{3.14}$$

where we assumed that the q_i vanish at $x = \pm\infty$. The linear system $\delta(\chi - \chi^{(0)}) = \lambda\, \mathrm{D}\chi$ reads

$$(\partial_t - \partial_x)\chi_1 = 0\,, \quad (\partial_t - \partial_x)\chi_i = -\lambda\, G_i^{-1} * G_{i-1} * \chi_{i-1} \qquad i = 2, \ldots, n \tag{3.15}$$

and

$$\begin{aligned} \chi_{i+1} - \chi_i &= \lambda\, (\partial_t + \partial_x + M_i *)\chi_i \qquad i = 1, \ldots, n-1 & (3.16) \\ \chi_n &= 1 - \lambda\, (\partial_t + \partial_x + M_n *)\chi_n\;. & (3.17) \end{aligned}$$

Using $\chi_i^{(0)} = 1$, (3.16) yields $\chi_{i+1}^{(1)} - \chi_i^{(1)} = M_i$ and thus[2] $\chi_i^{(1)} = -\sum_{k=i}^n M_k$. After some manipulations and using (3.10), we obtain

$$\begin{aligned} Q^{(2)} &= -\int dx \sum_{i=1}^n \sum_{k=i}^n (\partial_t M_k + M_i * M_k) \\ &= -\int dx \left(\sum_{i=1}^{n-1} e^{q_i - q_{i+1}} + \frac{1}{2} \left(\sum_{i=1}^n (\partial_t + \partial_x) q_i \right)^2 - \frac{1}{2} \sum_{i=1}^n [(\partial_t + \partial_x) q_i]^2 \right) \\ &\quad + \mathcal{O}(\theta) \end{aligned} \tag{3.18}$$

where to first order in θ already a rather complicated expression emerges. At 0th order in θ one recovers the known conserved charges of the Toda theory.

Infinite-dimensional integrable models possess an infinite set of conserved currents. In contrast to previous approaches to deformations of integrable models (see [6], for example), our approach guarantees, via deformation of the bicomplex associated with an integrable model, that this infinite tower of conservation laws survives the deformation.[3]

References

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[2] A "constant of integration" can be added on the rhs. But this would simply lead to an additional term proportional to $Q^{(1)}$ in (3.18).

[3] Deforming a Hamiltonian system which is (Liouville) integrable so that the conserved charges are in involution with respect to a symplectic structure, the question arises whether there is a corresponding deformation of the symplectic and Hamiltonian structure such that the deformation preserves the involution property.

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